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D-branes in $AdS_3 \times S^3 \times S^3 \times S^1$

Jose M Figueroa-O'Farrill

Department of Mathematics and Statistics, University of Edinburgh,
King's Buildings (JCMB), Mayfield Road, Edinburgh EH9 3JZ, Scotland
E-mail: jmf@maths.ed.ac.uk

Sonia Stanciu

Institute for Theoretical Physics, Utrecht University,
Princetonplein 5, 3584 CC Utrecht, The Netherlands
E-mail: s.stanciu@phys.uu.nl

Abstract: We analyse the possible D-brane configurations in an $AdS_3 \times S^3 \times S^3 \times S^1$ background with a NS-NS B field, by using the boundary state formalism and the classical open string sigma model. We study their geometry and we determine the fraction of space-time supersymmetry preserved by those solutions which admit a supersymmetric generalisation.

Keywords: D-branes, Conformal and W Symmetry, Differential and Algebraic Geometry.

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Contents

1. Introduction

In [1] we have initiated a study of the possible D-brane configurations in exact superstring backgrounds based on AdS spaces and characterized by a NS-NS antisymmetric field. Here we will continue this study by considering the case of the $AdS_3 \times S^3 \times S^3 \times S^1$ background. String propagation on this background was studied in [2]. This geometry appears as the throat limit of two differently oriented coincident sets of 5-branes intersecting in one direction, together with a set of infinitely stretched strings [3, 4, 5].

The paper is organized as follows. In section 2 we start with a short summary describing the bosonic background in order to set the notation and exhibit the conformal structure. In section 3 we consider the boundary state formalism adapted to this particular model; we write down the gluing conditions which preserve conformal invariance and the affine symmetry of the underlying current algebra and solve for them, thus determining two classes of bosonic configurations. In section 4 we identify the D-brane configurations that each of the two classes of solutions give rise to. The first class of solutions describe D-brane configurations which can be thought of as a straightforward generalisation of the D-type configurations in $AdS_3 \times S^3 \times T^4$ [1]. By contrast, the second class of solutions has no such analogue in the $AdS_3 \times S^3 \times T^4$

background. Therefore we devote section 5 to determining the twisted conjugacy classes of SU(2) $\times$ SU(2) and analysing their geometry and topology. In section 6 we extend our analysis to the $N = 1$ supersymmetric case and we find that all the bosonic configurations determined previously can be made into $N = 1$ configurations. We then analyse, in section 7, the fraction of supersymmetry preserved by these configurations, and we find that they preserve half of the space-time supersymmetry. We end, in section 8, with a summary of results.

2. The bosonic $AdS_3 \times S^3 \times S^3 \times S^1$ background

Bosonic string propagation on the background $AdS_3 \times S^3 \times S^3 \times S^1$ can be described using a free compact boson for the S^1 factor, and a WZW model with semisimple group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ for the $AdS_2 \times S^3 \times S^3$ part.

The theory of the compact free boson for the S^1 factor is described by the action

$$I_{S^1}[\varphi] = \int_\Sigma \partial\varphi\, \bar{\partial}\varphi\,,$$

where Σ is an orientable Riemann surface. The operator product algebra is standard:

$$\partial\varphi(z)\, \partial\varphi(w) = \frac{1}{(z-w)^2} + \mathrm{reg}\,, \tag{2.1}$$

with similar operator product expansions for the antiholomorphic sector.

The WZW model describing the $AdS^3 \times S^3 \times S^3$ part has as target the semisimple group $\mathbf{G} = \mathbf{G_1} \times \mathbf{G_2} \times \mathbf{G_3}$, with $\mathbf{G_1} = \mathrm{SL}(2,\mathbb{R})$, and $\mathbf{G_2}$ and $\mathbf{G_3}$ two copies of SU(2). The corresponding action will therefore be a sum of three terms

$$I = I_{\mathrm{SL}(2,\mathbb{R})}[g_1] + I_{\mathrm{SU}(2)}[g_2] + I_{\mathrm{SU}(2)}[g_3]\,, \tag{2.2}$$

where each term is of the standard form

$$\frac{2}{k_i} I[g_i] = \int_\Sigma \langle g_i^{-1}\partial g_i, g_i^{-1}\bar{\partial} g_i\rangle + \frac{1}{6}\int_B \langle g_i^{-1}dg_i, [g_i^{-1}dg_i, g_i^{-1}dg_i]\rangle\,,$$

with $\partial B = \Sigma$. Each of the fields g_i is a map from Σ to the Lie group $\mathbf{G_i}$, $i = 1, 2, 3$. We denote by $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ the corresponding Lie algebra, where $\mathfrak{g}_1 = \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{g}_2, \mathfrak{g}_3 = \mathfrak{su}(2)$. For these algebras we choose the following bases of generators: $\{T_a\}$ for $\mathfrak{g}_1$, $\{X_a\}$ for $\mathfrak{g}_2$ and $\{Y_a\}$ for $\mathfrak{g}_3$ satisfying

$$[T_1, T_2] = T_3\,, \qquad [T_2, T_3] = -T_1\,, \qquad [T_3, T_1] = -T_2\,,$$

and

$$[X_1, X_2] = X_3\,, \qquad [X_2, X_3] = X_1\,, \qquad [X_3, X_1] = X_2\,,$$

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and similar Lie brackets for the Y_a's. We also need to specify an invariant metric on $\mathfrak{g}$, which has a diagonal form

$$\eta = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix},$$

with three components given by $(\eta_1)_{ab} = \mathrm{diag}(+,+,-)$ and $(\eta_2)_{ab} = (\eta_3)_{ab} = \mathrm{diag}(+,+,+)$.

A group element $g = (g_1, g_2, g_3)$ in $\mathbf{G}$ can be parametrised as follows:

$$g_1 = e^{\theta_2 T_2} e^{\theta_1 T_1} e^{\theta_3 T_3}, \qquad g_2 = e^{\phi_2 X_2} e^{\phi_1 X_1} e^{\phi_3 X_3}, \tag{2.3}$$

$$g_3 = e^{\sigma_2 Y_2} e^{\sigma_1 Y_1} e^{\sigma_3 Y_3}, \tag{2.4}$$

where θ_μ, ϕ_μ and σ_μ, $\mu = 1,2,3$, play the role of the space-time fields. In terms of them (2.2) becomes a sigma-model action, whose space-time metric and antisymmetric field can be straightforwardly obtained (the expressions for $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ were explicitly written down in [1]).

This model possesses, as is well known, an infinite-dimensional symmetry group $\mathbf{G}(z) \times \mathbf{G}(\bar{z})$ characterized by the conserved currents $\mathbb{J}(z) = -\partial g g^{-1}$ and $\bar{\mathbb{J}}(\bar{z}) = g^{-1}\bar{\partial} g$, which underlies the exact conformal invariance of this background. The conserved currents generate, at the quantum level, an affine Lie algebra, $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2 \oplus \widehat{\mathfrak{g}}_3$, described by

$$\mathbb{J}_a(z)\mathbb{J}_b(w) = \frac{h_{ab}}{(z-w)^2} + \frac{f_{ab}{}^c \mathbb{J}_c(w)}{z-w} + \mathrm{reg}\,, \tag{2.5}$$

where the indices run over the whole Lie algebra, $a, b = 1, \ldots, 9$, and the coefficients h_{ab} define the symmetric bilinear form

$$h = \begin{pmatrix} k_1\eta_1 & 0 & 0 \\ 0 & k_2\eta_2 & 0 \\ 0 & 0 & k_3\eta_3 \end{pmatrix}. \tag{2.6}$$

The parameters k_i are related to the level x_i of the corresponding affine algebra by $k_i = x_i + g_i^*$, where g_i^* is the dual Coxeter number.

The CFT corresponding to this string background is then described by the energy-momentum tensor

$$\mathsf{T} = \Omega^{ab}(\mathbb{J}_a\mathbb{J}_b) + (\partial\varphi\partial\varphi)\,,$$

where Ω^{ab} are the components of the inverse Ω^{-1} of the invariant metric

$$\Omega = \begin{pmatrix} \Omega_1 & 0 & 0 \\ 0 & \Omega_2 & 0 \\ 0 & 0 & \Omega_3 \end{pmatrix},$$

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with the components given by $\Omega_1 = 2(k_1+1)\eta_1$, $\Omega_2 = 2(k_2-1)\eta_2$, and $\Omega_3 = 2(k_3-1)\eta_3$. The central charge of this CFT is given by

$$c = \frac{3k_1}{k_1+1} + \frac{3k_2}{k_2-1} + \frac{3k_3}{k_3-1} + 1\,.$$

3. Boundary states

The strategy that we will follow in order to determine the D-brane configurations which can be consistently defined in type IIB string theory on $AdS_3 \times S^3 \times S^3 \times S^1$ is similar to the one used in [1] and further developed in [6]. We consider a class of gluing conditions, which is defined in terms of a Lie algebra automorphism, $R : \mathfrak{g} \to \mathfrak{g}$, which preserves the metric η:

$$\big[R(Z_a), R(Z_b)\big] = R\big([Z_a, Z_b]\big)\,, \tag{3.1}$$

$$R^T \eta R = \eta\,, \tag{3.2}$$

where $\{Z_a\}$ is a given basis in $\mathfrak{g}$, in terms of which R is given by $R(Z_a) = Z_b R^b{}_a$. The gluing conditions read

$$\mathbb{J}_a(z) - R^b{}_a \bar{\mathbb{J}}_b(\bar{z}) = 0\,, \tag{3.3}$$

and can be easily seen to preserve the current algebra of the bulk theory.

These gluing conditions have to satisfy the basic consistency requirement, which is conformal invariance. In the bosonic case, this comes down to imposing

$$\mathsf{T}(z) = \bar{\mathsf{T}}(\bar{z})\,,$$

at the boundary. In this case, the requirement of conformal invariance translates into the condition

$$R^T \Omega R = \Omega\,. \tag{3.4}$$

As explained in [7], D-branes in a WZW model with group $\mathbf{G}$ are classified by the group $\mathrm{Out}_o(\mathbf{G})$ of metric-preserving outer automorphisms of $\mathbf{G}$, which is defined as the quotient $\mathrm{Aut}_o(\mathbf{G})/\mathrm{Inn}_o(\mathbf{G})$ of the group of metric-preserving automorphisms by the invariant subgroup of inner automorphisms. For the case at hand, and ignoring the S^1 factor for which no automorphism is inner, $\mathrm{Out}_o(\mathbf{G}) \cong \mathbb{Z}_2$, whence there are two distinct types of D-branes on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$. Let us see this.

Group automorphisms for simply connected groups come by exponentiating automorphisms of the Lie algebra. In our case, the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ is a direct sum of three terms $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_3$, each of them being a three-dimensional simple Lie algebra. Since $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ are non-isomorphic simple Lie algebras, there is no non-trivial homomorphism between them. We therefore deduce that the matrix of boundary conditions defined by the automorphism $R : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \to \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$, must take one of the following two forms:

$$R_I = \begin{pmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & 0 \\ 0 & 0 & R_{33} \end{pmatrix}, \qquad R_{II} = \begin{pmatrix} R_{11} & 0 & 0 \\ 0 & 0 & R_{23} \\ 0 & R_{32} & 0 \end{pmatrix}, \tag{3.5}$$

where $R_{ij} : \mathfrak{g}_j \to \mathfrak{g}_i$ for any $i, j = 1, 2, 3$. We thus have two classes of solutions: the first, described by R_I, exists for any values of the parameters k_i, whereas the second, given by R_{II}, exists only for particular values of the parameters, such that $k_2 = k_3$. Moreover, from (3.4) and (3.2), it follows that R_{11} belongs to $\mathrm{O}(2,1)$, and R_{ij}, with $i, j = 2, 3$, belong to $\mathrm{O}(3)$. On the other hand, from (3.1) we deduce that R_{ii} are Lie algebra automorphisms, corresponding to $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$, whereas R_{23} and R_{32} are Lie algebra isomorphisms. Explicitly, each of these conditions translates into a condition on the corresponding matrix, that reads

$$\det(R_{ij}) = 1\,,$$

which makes R_{11} belong to $\mathrm{SO}(2,1)$, and R_{ij}, for $i, j = 2, 3$, to $\mathrm{SO}(3)$.

Clearly, the main difference between these two classes of solutions is that R_I describe inner automorphisms, whereas R_{II} does not. More precisely, we have

$$R_{II} = T R_I\,,$$

where the matrix T is given by

$$T = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} \\ 0 & \mathbb{1} & 0 \end{pmatrix},$$

with obvious notation.

These results can be summarized as follows. We consider a set of gluing conditions on the group manifold $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ which preserve conformal invariance and the infinite-dimensional symmetry of the current algebra of the bulk theory. These gluing conditions are described in terms of metric-preserving automorphisms of the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. They admit two classes of solutions, characterized by different matrices of gluing conditions: for generic values of the levels k_i, the solutions are parametrised by elements of $\mathrm{SO}(2,1)\times\mathrm{SO}(3)\times\mathrm{SO}(3)$; additionally, for the particular values of k_i such that $k_2 = k_3$ we have an extra set of solutions, parametrised again by the elements of the group $\mathrm{SO}(2,1) \times \mathrm{SO}(3) \times \mathrm{SO}(3)$.

4. D-brane solutions

Let us begin with the D-brane configurations produced by the inner automorphisms R_I. This case represents a straightforward generalisation of the corresponding analysis performed for the $AdS_3 \times S^3 \times T^4$ background. Indeed, since the matrix of gluing conditions R_I is block diagonal, the resulting D-brane configurations take a product form, $\mathcal{D}_{\mathrm{SL}(2,\mathbb{R})} \times \mathcal{D}_{\mathrm{SU}(2)} \times \mathcal{D}_{\mathrm{SU}(2)}$, where $\mathcal{D}_{\mathbf{G}}$ represent the D-brane configurations in the group manifold $\mathbf{G}$. The possible D-brane configurations in $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ have been studied in detail in [1]. We therefore obtain in our case that the D-brane solution passing through a point g in $\mathbf{G}$ and being described by a set of gluing conditions defined by the inner automorphism R_I is characterized by a worldvolume which lies along a product of conjugacy classes shifted by group elements determined by R_I:

$$\mathcal{C}_{\mathrm{SL}(2,\mathbb{R})}\big(g_1 r_1^{-1}\big) r_1 \times \mathcal{C}_{\mathrm{SU}(2)}\big(g_2 r_2^{-1}\big) r_2 \times \mathcal{C}_{\mathrm{SU}(2)}\big(g_3 r_3^{-1}\big) r_3 \,,$$

where $R_{11} = \mathrm{Ad}_{r_1}$, $R_{22} = \mathrm{Ad}_{r_2}$, $R_{33} = \mathrm{Ad}_{r_3}$, for some (r_1, r_2, r_3) in $\mathbf{G}$. The conjugacy classes of $\mathrm{SU}(2)$ are parametrised by $S^1/\mathbb{Z}_2$, which we can understand as the interval $\theta \in [0, \pi]$. The conjugacy classes corresponding to $\theta = 0, \pi$ are points, corresponding to the elements $\pm e$ in the centre of $\mathrm{SU}(2)$, whereas the classes corresponding to $\theta \in (0, \pi)$ are 2-spheres. If we picture $\mathrm{SU}(2)$, which is homeomorphic to the 3-sphere, as the one-point compactification of $\mathbb{R}^3$ where the sphere at infinity is collapsed to a point, the foliation of $\mathrm{SU}(2)$ by its conjugacy classes coincides with the standard foliation of $\mathbb{R}^3$ by 2-spheres with two degenerate spheres at the origin and at infinity. For $\mathrm{SL}(2,\mathbb{R})$, on the other hand, we have three types of metrically nondegenerate[1] conjugacy classes (for details, see [1]): two point-like ones, corresponding to the two elements in the centre of $\mathrm{SL}(2,\mathbb{R})$, a family of two-dimensional classes with planar topology and a family of two-dimensional classes with cylindrical topology.

We now turn to the D-branes produced by outer automorphisms R_{II}. Notice first of all that, since the matrix of gluing conditions R_{II} has a block diagonal form, the resulting D-brane configurations take, also in this case, a product form, $\mathcal{D}_{\mathrm{SL}(2,\mathbb{R})} \times \mathcal{D}_{\mathrm{SU}(2)\times\mathrm{SU}(2)}$, where $\mathcal{D}_{\mathrm{SL}(2,\mathbb{R})}$ represent the same D-brane configurations in the group manifold $\mathrm{SL}(2,\mathbb{R})$ which were discussed above. It remains to analyse the $\mathcal{D}$-brane configurations on the product group $\mathrm{SU}(2) \times \mathrm{SU}(2)$, corresponding to the gluing conditions

$$-\partial\tilde{g}\tilde{g}^{-1} = \tilde{R}_{II}\big(\tilde{g}^{-1}\bar{\partial}\tilde{g}\big)\,, \tag{4.1}$$

where $\tilde{g} = (g_2, g_3)$, and the corresponding matrix of gluing conditions

$$\tilde{R}_{II} = \begin{pmatrix} 0 & R_{23} \\ R_{32} & 0 \end{pmatrix}$$

can be easily shown to be of the form

$$\tilde{R}_{II} = \tilde{T}\,\mathrm{Ad}_{\tilde{r}}\,, \qquad \tilde{T} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},$$

[1] This condition is necessary for their interpretation as D-branes.

where $\tilde{r} = (r_2, r_3)$, while $R_{32} = \mathrm{Ad}_{r_2}$ and $R_{23} = \mathrm{Ad}_{r_3}$. This implies that it is sufficient to analyse the D-brane configurations produced by $\tilde{R}_{II} = \tilde{T}$ in $\mathrm{SU}(2) \times \mathrm{SU}(2)$; any other $\tilde{R}_{II}$ will lead to configurations which differ from these only by translations in the group manifold. Indeed, a set of gluing conditions described by a generic $\tilde{R}_{II}$

$$-\partial \tilde{g} \tilde{g}^{-1} = \tilde{T} \cdot \mathrm{Ad}_{\tilde{r}} \left(\tilde{g}^{-1} \bar{\partial} \tilde{g} \right) ,$$

for some $\tilde{r}$ in $\mathrm{SU}(2) \times \mathrm{SU}(2)$, can be written as

$$-\partial h h^{-1} = \tilde{T} \big(h^{-1} \bar{\partial} h \big) ,$$

with $h = \tilde{g}\tilde{r}^{-1}$. According to the general theory developed in [6] the D-brane configurations produced by the gluing conditions (4.1) with $\tilde{R}_{II} = \tilde{T}$ are nothing but the twisted conjugacy classes determined by the outer automorphism $\tilde{T}$, which we denote by $\mathcal{C}^{\tilde{T}}_{\mathrm{SU}(2)\times\mathrm{SU}(2)}(\tilde{g})$. In summary, the D-brane configurations passing through a point g in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ and described by the matrix of gluing conditions R_{II} have worldvolumes which lie along a product of twisted conjugacy classes

$$\mathcal{C}_{\mathrm{SL}(2,\mathbb{R})} \big(g_1 r_1^{-1} \big) r_1 \times \mathcal{C}^{\tilde{T}}_{\mathrm{SU}(2)\times\mathrm{SU}(2)} \big(\tilde{g} \tilde{r}^{-1} \big) \tilde{r} \, .$$

Thus, in order to complete our analysis, we have to determine the twisted conjugacy classes of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Next section will be devoted to solving this mathematical problem. There we will show that the twisted conjugacy classes of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ corresponding to the outer automorphism $\tilde{T}$ are basically determined by the (ordinary) conjugacy classes of $\mathrm{SU}(2)$. Indeed, if we denote by m the group multiplication, $m : \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SU}(2)$, which assigns to every element (g_2, g_3) in $\mathrm{SU}(2) \times \mathrm{SU}(2)$ an element in $\mathrm{SU}(2)$ given by $m(g_2, g_3) = g_2 g_3$, then we have that

$$\mathcal{C}^{\tilde{T}}_{\mathrm{SU}(2)\times\mathrm{SU}(2)}(g_2, g_3) = m^{-1} \mathcal{C}_{\mathrm{SU}(2)}(g_2 g_3) \, .$$

Using this result we will see that the group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ has two types of conjugacy classes: two three-dimensional ones, diffeomorphic to S^3, and a family of five-dimensional classes, diffeomorphic to $S^2 \times S^3$.

5. Some twisted conjugacy classes in $\mathrm{SU}(2) \times \mathrm{SU}(2)$

The primary goal of this section is to determine the twisted conjugacy classes of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and understand their geometry and topology. However we think it might be useful to consider in the beginning a slightly more general problem, which basically consists in replacing $\mathrm{SU}(2)$ with an arbitrary group.

Let G be a Lie group and let $D = G \times G$ be the product group. Let $\tau : D \to D$ be the twist $\tau(x, y) = (y, x)$. It is clearly an outer automorphism of D.

Let Ad_τ denote the twisted adjoint action of D on D,

$$\mathrm{Ad}_\tau(x,y)\cdot(x_0,y_0) = \left(xx_0y^{-1}, yy_0x^{-1}\right),$$

for all $(x,y),(x_0,y_0)\in D$. By the twisted conjugacy class of a point $(x,y)\in D$, we mean the orbit of (x,y) under the twisted adjoint action of D on D. We denote it $\mathcal{O}_{(x,y)}$. In other words,

$$\mathcal{O}_{(x,y)} = \left\{\left(wxz^{-1}, zyw^{-1}\right) \mid w,z\in G\right\}.$$

In the notation of the last section, $\mathcal{O}_{(x,y)} = \mathcal{C}^{\tilde{T}}_{G\times G}(x,y)$.

We would like to determine the twisted conjugacy classes of D and explore their topology. If, as in the case of interest, G (and hence D) possesses a bi-invariant metric, we also would like to say something about the geometry of the twisted conjugacy classes as submanifolds of D.

We start with an example. The twisted conjugacy class of the identity (e,e) is the "anti-diagonal", a submanifold of D diffeomorphic to G, but which is not a subgroup:

$$\mathrm{Ad}_\tau(x,y)\cdot(e,e) = \left(xy^{-1}, yx^{-1}\right) = \left(z, z^{-1}\right)$$

for $z := xy^{-1}$. Hence the orbit $\mathcal{O}_{(e,e)}$ of (e,e) is

$$\mathcal{O}_{(e,e)} = \left\{\left(z,z^{-1}\right) \mid z\in G\right\} \cong G,$$

where the isomorphism is one of differentiable manifolds.

This example points the way to determining the rest of the twisted conjugacy classes. We start with a series of observations.

Group multiplication gives a natural surjection $m : D = G\times G\to G$. We will write it simply as $m(x,y) = xy$. It is easy to see that the inverse image of a point is diffeomorphic to G. Indeed, suppose $xy = x'y'$. Then $x^{-1}x' = y(y')^{-1} = z$, say. Therefore, $x' = xz$ and $y' = z^{-1}y$, for some $z\in G$. In other words,

$$m^{-1}(xy) = \left\{\left(xz, z^{-1}y\right) \mid z\in G\right\} \cong G.$$

This means that $G\times G$ is a bundle over G with fibre G, but the fibration is different than the standard fibrations $\mathrm{pr}_1 : G\times G\to G$ sending $(x,y)\mapsto x$ and $\mathrm{pr}_2 : G\times G\to G$ sending $(x,y)\mapsto y$. Nevertheless $m : G\times G\to G$ is a principal G-bundle. To prove this we must simply exhibit a free action of G which preserves the fibres. The typical fibre is given by

$$m^{-1}(xy) = \left\{\left(xz, z^{-1}y\right) \mid z\in G\right\}.$$

Let $g\in G$ and consider the action $(x,y)\mapsto(xg^{-1}, gy)$. This is clearly free and moreover $xg^{-1}gy = xy$, whence it preserves the fibre.

We can now state the main result of this section.

Theorem 1. *Let $\mathcal{O}_{(x,y)}$ be the twisted conjugacy class of $(x,y) \in D$ and let $\mathcal{C}_z$ be the (standard) conjugacy class of $z \in G$. Then,*

$$\mathcal{O}_{(x,y)} = m^{-1}\mathcal{C}_{xy}\,.$$

In other words, twisted conjugacy classes in D are the inverse images under the group multiplication of the (standard) conjugacy classes in G.

Proof. A typical element in $\mathcal{O}_{(x,y)}$ is (uxv^{-1}, vyu^{-1}). The product of these two elements is $uxyu^{-1}$, which is conjugate to $xy \in G$. In other words, $m(\mathcal{O}_{(x,y)}) = \mathcal{C}_{xy}$, whence $\mathcal{O}_{(x,y)} \subseteq m^{-1}\mathcal{C}_{xy}$. To prove the reverse inclusion, we will prove that if ab and cd are conjugate in G, then (a,b) and (c,d) belong to the same twisted conjugacy class in D. If ab and cd are conjugate, then $cd = zabz^{-1}$ for some $z \in G$. This is equivalent to

$$\begin{aligned} z^{-1}cd = abz^{-1} &\iff a^{-1}z^{-1}c = bz^{-1}d^{-1} = w^{-1} \quad \exists w \in G \\ &\iff c = zaw^{-1} \quad \text{and} \quad d = wbz^{-1} \\ &\iff (c,d) = \left(zaw^{-1}, wbz^{-1}\right), \end{aligned}$$

whence (c,d) and (a,b) are in the same twisted conjugacy class. $\square$

As a corollary we have that twisted conjugacy classes in $D = G \times G$ are principal G-bundles over conjugacy classes of G.

We now specialize to $G = \mathrm{SU}(2)$. The Lie group $\mathrm{SU}(2)$ has two kinds of conjugacy classes: points (at $\pm e$) and 2-spheres everywhere else. Now, the inverse image under m of the points are topologically 3-spheres, whereas the inverse image of a 2-sphere is a principal $\mathrm{SU}(2)$-bundle over S^2. Since principal G-bundles over S^2 are classified up to homotopy by $\pi_1(G)$, the fact that $\mathrm{SU}(2)$ is simply-connected implies that the bundle is trivial. In other words, the Lie group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ has two types of twisted conjugacy classes, diffeomorphic to S^3 or to $S^2 \times S^3$.

Notice furthermore that the S^3 orbits are homologically non trivial. This is because the maps induced in homology by the canonical projections pr_1 and pr_2 send the homology classes of the orbits to the fundamental class of $\mathrm{SU}(2)$, up to orientation.

For example, if $\mathcal{O} = \mathcal{O}_{(e,e)}$, notice that

$$\mathrm{pr}_1\left(z, z^{-1}\right) = z \qquad \text{and} \qquad \mathrm{pr}_2\left(z, z^{-1}\right) = z^{-1}\,.$$

Therefore $(\mathrm{pr}_1)_*[\mathcal{O}] = 1 \in \mathbb{Z} \cong H_3(S^3)$ and $(\mathrm{pr}_2)_*[\mathcal{O}] = -1 \in \mathbb{Z} \cong H_3(S^3)$.

On the other hand, the five-dimensional classes are homologically trivial, since $H_5(S^3 \times S^3) = 0$. Nevertheless a similar argument shows that they are not homotopically trivial.

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It remains to see how the twisted conjugacy classes are embedded geometrically relative to the bi-invariant metric on $G \times G$. We will first consider the small orbits (of dimension $\dim G$) obtained from point-like conjugacy classes in G. As we will show, they are totally geodesic submanifold of D; and, in particular, they are minimal.

Consider $\mathcal{O} = \mathcal{O}_{(e,e)}$. It is not a subgroup of D, but we will see that it is a subgroup of D', a Lie group which shares the same underlying manifold as D, but whose group multiplication is different. Moreover, the bi-invariant metric on D is also bi-invariant in D'. It will follow that $\mathcal{O}$ is totally geodesic as a consequence of the following well-known result (see, for example, exercise 6.6 in [8]):

Theorem 2. *Let G be a Lie group with a bi-invariant metric. Then any subgroup H is a totally geodesic submanifold.*

The group D' is defined as $G \times G^{\mathrm{opp}}$, where G^{opp}, the opposite group, is the group sharing the same underlying manifold with G but with the opposite multiplication law:

$$\begin{aligned} m^{\mathrm{opp}} : G^{\mathrm{opp}} \times G^{\mathrm{opp}} &\longrightarrow G^{\mathrm{opp}} \\ (x,y) &\longmapsto yx\,. \end{aligned}$$

In other words, $m^{\mathrm{opp}} = m \circ \tau$. Clearly G^{opp} and G are isomorphic as Lie groups: the isomorphism $G \to G^{\mathrm{opp}}$ being defined by $x \mapsto x^{-1}$.

Under the group multiplication in $D' = G \times G^{\mathrm{opp}}$,

$$\begin{aligned} m' : D' \times D' &\longrightarrow D' \\ \big((x,y),(u,v)\big) &\longmapsto (xu, vy)\,, \end{aligned}$$

it is clear that $\mathcal{O}$ is a subgroup:

$$\big(x, x^{-1}\big) \cdot \big(y, y^{-1}\big) = \big(xy, y^{-1}x^{-1}\big) = \big(xy, (xy)^{-1}\big)\,.$$

Since D' has the same underlying manifold as D, we have a D-bi-invariant metric on it. To be able to apply the theorem, it remains to show that this metric is also D'-bi-invariant. The bi-invariant metric on $D = G \times G$ is the riemannian product of the *same* bi-invariant metric on each of the factors: this guarantees that τ is an isometry. Therefore this metric on $G \times G$ will be bi-invariant under D' if and only if the metric on G is G^{opp}-bi-invariant. But this follows trivially from the following observation. For $x \in G$ let $L(x)$ and $R(x)$ denote the left- and right-multiplication by x in G, respectively. Similarly, let $L^{\mathrm{opp}}(x)$ and $R^{\mathrm{opp}}(x)$ denote the similar operations in G^{opp}. Then one has

$$L(x) = R^{\mathrm{opp}}(x) \qquad \text{and} \qquad R(x) = L^{\mathrm{opp}}(x)\,.$$

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This means that left-invariance under G is equivalent to right-invariance under G^{opp} and vice versa. In particular, bi-invariance under G is equivalent to bi-invariance under G^{opp}. We conclude that since the metric on G is G-bi-invariant, it is also G^{opp}-bi-invariant. In summary, we have just proven the following.

Theorem 3. *The twisted conjugacy class $\mathcal{O}_{(e,e)}$ is totally-geodesic relative to the bi-invariant metric on D. In particular, it is minimal.*

How about the other small twisted conjugacy classes? These are the inverse images by the multiplication m of point-like conjugacy classes in G, hence of elements in the centre of G. Let z be an element in the centre of G. Then the twisted conjugacy class $\mathcal{O}_{(e,z)} := m^{-1}(z)$ is given by

$$\mathcal{O}_{(e,z)} = \left\{ \left(x, x^{-1}z\right) \mid x \in G \right\} .$$

This is the translate (both left and right) of $\mathcal{O}_{(e,e)}$ by the element (e, z). Since the metric on D is bi-invariant, $\mathcal{O}_{(e,z)}$ is isometric to $\mathcal{O}_{(e,e)}$ as submanifolds of D. In particular, since $\mathcal{O}_{(e,e)}$ is totally geodesic, so is $\mathcal{O}_{(e,z)}$. This proves the following.

Theorem 4. *Let $z \in G$ be any element in the centre. The twisted conjugacy class $\mathcal{O}_{(e,z)}$ is totally-geodesic relative to the bi-invariant metric on D. In particular, it is minimal.*

Specializing to $G = \mathrm{SU}(2)$ we have that the twisted conjugacy classes $\mathcal{O}_{(e,e)}$ and $\mathcal{O}_{(e,-e)}$ in $\mathrm{SU}(2) \times \mathrm{SU}(2)$ are totally-geodesic 3-spheres.

How about the larger orbits? In this case, it is possible to argue that it is metrically a fibre product with totally-geodesic fibres diffeomorphic to G. But the total space of the bundle is certainly not totally geodesic. In fact, it is in general not even minimal.

6. The $N = 1$ supersymmetric extension

Let us now consider the $N = 1$ supersymmetric extension of the affine Lie algebra $\widehat{\mathfrak{g}}$, which we will denote by $\widehat{\mathfrak{g}}_{N=1} = \widehat{\mathfrak{sl}}(2,\mathbb{R})_{N=1} \oplus \widehat{\mathfrak{su}}(2)_{N=1}$, with generators $(\mathbb{J}_a, \Psi_a)$ satisfying

$$\mathbb{J}_a(z)\mathbb{J}_b(w) = \frac{h_{ab}}{(z-w)^2} + \frac{f_{ab}{}^c\mathbb{J}_c(w)}{z-w} + \text{reg}\,, \tag{6.1}$$

$$\mathbb{J}_a(z)\Psi_b(w) = \frac{f_{ab}{}^c\Psi_c(w)}{z-w} + \text{reg}\,, \tag{6.2}$$

$$\Psi_a(z)\Psi_b(w) = \frac{h_{ab}}{z-w} + \text{reg}\,, \tag{6.3}$$

with h_{ab} defined as in (2.6). The free fields (φ, λ) on S^1 satisfy the standard OPEs

$$\partial\varphi(z)\partial\varphi(w) = \frac{1}{(z-w)^2} + \mathrm{reg}\,, \tag{6.4}$$

$$\lambda(z)\lambda(w) = \frac{1}{z-w} + \mathrm{reg}\ . \tag{6.5}$$

Then the generators of the $N = 1$ SCA will be given by

$$\begin{aligned} \mathsf{T}(z) &= \frac{1}{2}h^{ab}\big(\tilde{\mathbb{J}}_a\tilde{\mathbb{J}}_b\big) + \frac{1}{2}h^{ab}(\partial\Psi_a\Psi_b) + \frac{1}{2}(\partial\varphi\partial\varphi) + \frac{1}{2}(\partial\lambda\lambda)\,, \\ \mathsf{G}(z) &= h^{ab}\big(\tilde{\mathbb{J}}_a\Psi_b\big) + (\partial\varphi\lambda) - \frac{1}{6k^2}f^{abc}(\Psi_a\Psi_b\Psi_c)\,, \end{aligned}$$

where we have introduced the so-called decoupled currents, $\tilde{\mathbb{J}}_a \equiv \mathbb{J}_a - \frac{1}{2}h^{bd}f_{ab}{}^c(\Psi_c\Psi_d)$, in terms of which the superconformal generators take a relatively simple form. The coefficients h^{ab} are the components of h^{-1}. The central charge of this SCFT is given by

$$c = \frac{3}{2} + \frac{3(k_1-1)}{k_1} + \frac{3}{2} + \frac{3(k_2+1)}{k_2} + \frac{3}{2} + \frac{3(k_3+1)}{k_3} + \frac{3}{2}\,.$$

In order to have a critical superstring theory the levels must satisfy the following relation:

$$\frac{1}{k_1} - \frac{1}{k_2} - \frac{1}{k_3} = 0\,,$$

in which case $c = 15$. And, since we have a similar structure for the antiholomorphic sector as well, we actually have a $(1,1)$ SCFT.

The gluing conditions are given by

$$\mathbb{J}_a(z) - R^b{}_a\bar{\mathbb{J}}_b(\bar{z}) = 0\,, \qquad \Psi_a(z) - S^b{}_a\bar{\Psi}_b(\bar{z}) = 0\,, \tag{6.6}$$

where the coefficients $R^b{}_a$ and $S^b{}_a$ are defined by $R, S : \mathfrak{g} \to \mathfrak{g}$, with $R(Z_a) = Z_bR^b{}_a$ and $S(Z_a) = Z_bS^b{}_a$, for any Z_a in $\mathfrak{g}$. These conditions are to be understood as supersymmetric generalizations of the gluing conditions written down in section 3; henceforth R is taken to be an automorphism of $\mathfrak{g}$ which preserves the metric. At this point we do not need to impose any specific condition on S, since this will be fixed, as we will see in a moment, by supersymmetry considerations.

The gluing conditions (6.6) have to satisfy a similar consistency requirement as in the bosonic case. In this context, consistency means that the holomorphic SCFT is set equal to the antiholomorphic SCFT up to an automorphism of the $N = 1$ SCA; in other words, at the boundary we must have

$$\mathsf{T}(z) = \bar{\mathsf{T}}(\bar{z}) \qquad \text{and} \qquad \mathsf{G}(z) = \pm\bar{\mathsf{G}}(\bar{z})\,.$$

These conditions have been written down previously in [9], in the context of Kazama-Suzuki models.

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The first requirement translates into a number of conditions on the matrices R and S. Thus, from the quadratic terms in the currents we obtain that

$$R^T \eta R = \eta \,, \qquad S^T \eta R = \pm \eta \,,$$

which immediately implies that

$$S = \pm R \,, \tag{6.7}$$

as one would expect from supersymmetry. Further, from the cubic terms in the currents we have that

$$\big[S(Z_a), S(Z_b)\big] = \pm S\big([Z_a, Z_b]\big) \,, \qquad \big[R(Z_a), S(Z_b)\big] = S\big([Z_a, Z_b]\big) \,,$$

which, together with (6.7), implies that

$$\big[R(Z_a), R(Z_b)\big] = R\big([Z_a, Z_b]\big) \,. \tag{6.8}$$

In other words, the conditions that R must satisfy in order for the corresponding D-brane configurations to preserve superconformal invariance match exactly the assumptions already made on R. Furthermore, it follows that these gluing conditions preserve the infinite-dimensional symmetry of the $N=1$ current algebra (6.1)–(6.3).

Finally, since we know from the bosonic case that R must take one of two particular forms (3.5), we obtain that S must have a similar form

$$S_I = \pm \begin{pmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & 0 \\ 0 & 0 & R_{33} \end{pmatrix} , \qquad S_{II} = \pm \begin{pmatrix} R_{11} & 0 & 0 \\ 0 & 0 & R_{23} \\ 0 & R_{32} & 0 \end{pmatrix} , \tag{6.9}$$

We therefore conclude that every bosonic configuration that we determined can be made into an $N=1$ supersymmetric configuration without having to impose additional conditions.

7. Space-time supersymmetry

In this section we analyse the fraction of space-time supersymmetry preserved by the D-brane configurations we determined before. In the context of superconformal field theories space-time supersymmetry appears as a by-product of $N=2$ superconformal invariance, being related, via bosonisation, to the U(1) current. Instead of following this standard approach, here we will analyse the space-time symmetry preserved by the D-branes we found using a different route, which was described in [10, 2].

We will therefore consider the space-time supercharges to be constructed directly from the $N=1$ SCFT. To this end we introduce the fermionic fields ψ_i for $\mathrm{SL}(2,\mathbb{R})$,

χ_i and ω_i for the two copies of SU(2), with $i = 1, 2, 3$. Further, we choose five fermion bilinears and bosonise them into five scalar fields H_I, with $I = 1, \ldots, 5$ as follows:

$$\partial H_1 = \psi_1 \psi_2 \,, \qquad \partial H_2 = \chi_1 \chi_2 \,, \qquad \partial H_3 = \omega_1 \omega_2 \,,$$
$$\partial H_4 = i \left(\sqrt{\frac{k_1}{k_2}} \chi_3 + \sqrt{\frac{k_1}{k_3}} \omega_3 \right) \psi_3 \,, \qquad \partial H_5 = \left(\sqrt{\frac{k_1}{k_2}} \chi_3 - \sqrt{\frac{k_1}{k_3}} \omega_3 \right) \lambda \,.$$

The corresponding space-time supercharges [11] will be required to be BRST invariant and to pass the GSO projection. This will yield

$$Q = \oint dz e^{-\phi/2} S(z) \,, \tag{7.1}$$

where ϕ is the scalar field which appears in the bosonised superghost system of the fermionic string, and the corresponding spin fields are given by (for a detailed discussion see [2])

$$\begin{aligned}
S_1(z) &= e^{i/2(H_1+H_2+H_3+H_4+H_5)} \,, \\
S_2(z) &= e^{i/2(H_1-H_2-H_3-H_4-H_5)} \,, \\
S_3(z) &= e^{i/2(-H_1+H_2+H_3-H_4+H_5)} \,, \\
S_4(z) &= e^{i/2(-H_1-H_2-H_3+H_4-H_5)} \,, \\
S_5(z) &= \sqrt{\frac{k_1}{k_2}} e^{i/2(H_1-H_2+H_3-H_4+H_5)} + \sqrt{\frac{k_1}{k_3}} e^{i/2(H_1-H_2+H_3+H_4-H_5)} \,, \\
S_6(z) &= \sqrt{\frac{k_1}{k_2}} e^{i/2(H_1+H_2-H_3+H_4-H_5)} + \sqrt{\frac{k_1}{k_3}} e^{i/2(H_1+H_2-H_3-H_4+H_5)} \,, \\
S_7(z) &= \sqrt{\frac{k_1}{k_2}} e^{i/2(-H_1-H_2+H_3+H_4+H_5)} + \sqrt{\frac{k_1}{k_3}} e^{i/2(-H_1-H_2+H_3-H_4-H_5)} \,, \\
S_8(z) &= \sqrt{\frac{k_1}{k_2}} e^{i/2(-H_1+H_2-H_3-H_4-H_5)} + \sqrt{\frac{k_1}{k_3}} e^{i/2(-H_1+H_2-H_3+H_4+H_5)} \,.
\end{aligned}$$

For any given D-brane configuration, the boundary conditions satisfied by the fermionic fields will lead to a certain set of boundary conditions satisfied by the supercharges. Let us consider the those configurations described by R_I, with R_{11} of the form of a spatial rotation. The corresponding boundary conditions satisfied by the fermions read (we use the notation introduced in [1])

$$\begin{aligned}
\psi_1 - \cos\alpha \bar{\psi}_1 - \sin\alpha \bar{\psi}_2 &= 0 \,, & \chi_1 - \cos\beta \bar{\chi}_1 - \sin\beta \bar{\chi}_2 &= 0 \,, \\
\psi_2 + \sin\alpha \bar{\psi}_1 - \cos\alpha \bar{\psi}_2 &= 0 \,, & \chi_2 + \sin\beta \bar{\chi}_1 - \cos\beta \bar{\chi}_2 &= 0 \,, \\
\psi_3 - \bar{\psi}_3 &= 0 \,, & \chi_3 - \bar{\chi}_3 &= 0 \,, \\
\omega_1 - \cos\gamma \bar{\omega}_1 - \sin\gamma \bar{\omega}_2 &= 0 \,, & & \\
\omega_2 + \sin\gamma \bar{\omega}_1 - \cos\gamma \bar{\omega}_2 &= 0 \,, & \lambda \pm \bar{\lambda} &= 0 \,, \\
\omega_3 - \bar{\omega}_3 &= 0 \,, & &
\end{aligned}$$

where we have systematically ignored a $\pm$ sign coming from (6.9), which does not affect the fermion bilinears in the expression of $S(z)$. The $\pm$ sign in the boundary condition on λ corresponds to having Neumann or Dirichlet boundary conditions on S^1, respectively. We will see, however, that only one choice will give rise to states preserving some space-time supersymmetry. From the above relations it follows that the scalar fields H_I satisfy the following boundary conditions:

$$H_I = \bar{H}_I\,, \quad I = 1,2,3,4\,, \qquad H_5 = \pm\bar{H}_5\,,$$

where the $\pm$ sign in the boundary condition for H_5 reflects the one in the boundary condition along S^1. Therefore, in order to preserve some space-time supersymmetry, we have to impose a Dirichlet boundary condition along the flat direction of the target space. In this way we obtain that these configurations describe odd-dimensional D-branes (as we expect in the case of a type IIB theory). It immediately follows that, for these configurations, the space-time supercharges satisfy

$$Q_\alpha = \bar{Q}_\alpha\,, \quad \alpha = 1,\ldots,8\,. \tag{7.2}$$

Hence we conclude that the D-brane configurations defined by R_I, with R_{11} a spatial rotation in SO(2,1), and characterized by a Dirichlet condition along S^1 preserve half of the space-time supersymmetry, and therefore are BPS states.

In order to analyse the configurations described by R_I with R_{11} given by a boost in SO(2,1) we need a slight change in the way we define the fermion bilinears and the corresponding supercharges. Essentially, we need to switch the places of ψ_2 and ψ_3 in the definition of H_1 and H_4. Then by using the following boundary conditions for the fermions on AdS_3

$$\begin{aligned}\psi_1 - \cosh\alpha\bar{\psi}_1 - \sinh\alpha\bar{\psi}_3 &= 0\,,\\ \psi_2 - \bar{\psi}_2 &= 0\,,\\ \psi_3 - \sinh\alpha\bar{\psi}_1 - \cosh\alpha\bar{\psi}_3 &= 0\,,\end{aligned}$$

and unchanged conditions for all the other fermions, we obtain that $H_I = \bar{H}_I$, for all I, provided we set again a Dirichlet boundary condition along S^1. From this it follows that the space-time supercharges satisfy (7.2) as before. Hence all the corresponding odd-dimensional D-branes describe BPS states that preserve half of the space-time supersymmetry.

Finally, in the case where R_{11} is given by a null rotation in SO(2,1), due to the particular form of the boundary conditions and to the non-local nature of the dependence of the space-time supercharges on the fermionic fields, it is rather difficult to determine the fraction of space-time supersymmetry preserved by this particular type of boundary states.

Let us now turn to the D-brane configurations described by R_{II}. If we start with configurations characterized by a component R_{11} of the form of a spatial rotation in

SO(2,1), then the corresponding boundary conditions on the fermions read

$$\begin{aligned}
\psi_1 - \cos\alpha\bar\psi_1 - \sin\alpha\bar\psi_2 &= 0\,, &\qquad \chi_1 - \cos\beta\bar\omega_1 - \sin\beta\bar\omega_2 = 0\,,\\
\psi_2 + \sin\alpha\bar\psi_1 - \cos\alpha\bar\psi_2 &= 0\,, &\qquad \chi_2 + \sin\beta\bar\omega_1 - \cos\beta\bar\omega_2 = 0\,,\\
\psi_3 - \bar\psi_3 &= 0\,, &\qquad \chi_3 - \bar\omega_3 = 0\,,\\
\omega_1 - \cos\gamma\bar\chi_1 - \sin\gamma\bar\chi_2 &= 0\,, &\\
\omega_2 + \sin\gamma\bar\chi_1 - \cos\gamma\bar\chi_2 &= 0\,, &\qquad \lambda \pm \bar\lambda = 0\,,\\
& \omega_3 - \bar\chi_3 = 0\,. &
\end{aligned}$$

This time, in order to be able to preserve some fraction of the space-time supersymmetry, we must impose a Neumann boundary condition along S^1. Then we obtain

$$H_1 = \bar H_1\,, \qquad H_2 = \bar H_3\,, \qquad H_3 = \bar H_2\,, \qquad H_4 = \bar H_4\,, \qquad H_5 = \bar H_5\,.$$

This in turn implies that the corresponding supercharges (where, we recall, $k_2 = k_3$) will satisfy the following conditions:

$$Q_\alpha = A^\beta{}_\alpha \bar Q_\beta, \qquad \alpha,\beta = 1,\ldots,8\,, \tag{7.3}$$

where the coefficients $A^\beta{}_\alpha$ are the elements of the matrix A

$$A = \begin{pmatrix} \mathbb{1} & & & \\ & \mathbb{1} & & \\ & & \sigma & \\ & & & \sigma \end{pmatrix},$$

where $\sigma = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathbb{1}$ is the two-dimensional identity matrix. This means that these configurations too preserve half the space-time supersymmetry and thus constitute BPS states.

Similar results hold in the case where the component R_{11} of R_{II} is given by a boost, provided we use the appropriate choice for the scalar fields H_I. And also similarly, we cannot say much about the boundary states having the R_{11} of the form of a null rotation.

The results are summarized in table 1.

8. Conclusions

In this paper we have studied, using the SCFT framework and the boundary state formalism, the possible D-brane configurations which one can consistently define in an $AdS_3 \times S^3 \times S^3 \times S^1$ background characterized by a purely NS-NS B field.

We have analyzed a certain type of gluing conditions (type-D, according to the nomenclature used in [6, 1]), which are characterized by the fact that they preserve

D7	$(\mathbb{R}\times S^1)\times S^2\times S^3\times S^1$ $\mathbb{R}^2\times S^2\times S^3\times S^1$	$H_I=\bar{H}_I,\ I=1,4,5$ $H_2=\bar{H}_3,\ H_3=\bar{H}_2$	$Q_\alpha = A^\beta{}_\alpha \bar{Q}_\beta$ $\alpha,\beta=1,\ldots,8$
D5	$(\mathbb{R}\times S^1)\times S^2\times S^2$ $\mathbb{R}^2\times S^2\times S^2$	$H_I=\bar{H}_I,\ I=1,\ldots,5$	$Q_\alpha=\bar{Q}_\alpha,\ \alpha=1,\ldots,8$
D5	$\{\pm e\}\times S^2\times S^3\times S^1$ $(\mathbb{R}\times S^1)\times\{\pm e\}\times S^3\times S^1$ $\mathbb{R}^2\times\{\pm e\}\times S^3\times S^1$	$H_I=\bar{H}_I,\ I=1,4,5$ $H_2=\bar{H}_3,\ H_3=\bar{H}_2$	$Q_\alpha = A^\beta{}_\alpha \bar{Q}_\beta,$ $\alpha,\beta=1,\ldots,8$
D3	$\{\pm e\}\times S^2\times S^2$ $(\mathbb{R}\times S^1)\times\{\pm e\}\times S^2$ $\mathbb{R}^2\times\{\pm e\}\times S^2$	$H_I=\bar{H}_I,\ I=1,\ldots,5$	$Q_\alpha=\bar{Q}_\alpha,\ \alpha=1,\ldots,8$
D3	$\{\pm e\}\times\{\pm e\}\times S^3\times S^1$	$H_I=\bar{H}_I,\ I=1,4,5$ $H_2=\bar{H}_3,\ H_3=\bar{H}_2$	$Q_\alpha = A^\beta{}_\alpha \bar{Q}_\beta,$ $\alpha,\beta=1,\ldots,8$
D1	$\{\pm e\}\times\{\pm e\}\times S^2$ $(\mathbb{R}\times S^1)\times\{\pm e\}\times\{\pm e\}$ $\mathbb{R}^2\times\{\pm e\}\times\{\pm e\}$	$H_I=\bar{H}_I,\ I=1,\ldots,5$	$Q_\alpha=\bar{Q}_\alpha,\ \alpha=1,\ldots,8$
D(-1)	$\{\pm e\}\times\{\pm e\}\times\{\pm e\}$	$H_I=\bar{H}_I,\ I=1,\ldots,5$	$Q_\alpha=\bar{Q}_\alpha,\ \alpha=1,\ldots,8$

Table 1: Space-time supersymmetry for D-brane configurations in $AdS_3\times S^3\times S^3\times S^1$.

not only the superconformal structure of the background but also the underlying symmetry of the $N=1$ current algebra. We have seen that the solutions fall in two different classes. The first class, produced by gluing conditions defined in terms of inner automorphisms of the corresponding Lie algebra, describes D-brane configurations whose worldvolumes are products of shifted conjugacy classes

$$\mathcal{C}_{\mathrm{SL}(2,\mathbb{R})}r_1\times\mathcal{C}_{\mathrm{SU}(2)}r_2\times\mathcal{C}_{\mathrm{SU}(2)}r_3\,,$$

giving thus rise to odd-dimensional D-branes embedded in $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)\times\mathrm{SU}(2)$ and even-dimensional D-branes wrapped on the flat S^1. It is however the odd-dimensional D-branes that turn out to also preserve half of the space-time supersymmetry of the background.

The second class of solutions, produced by gluing conditions defined in terms of outer automorphisms, describes D-brane configurations whose worldvolumes are products of shifted conjugacy classes in $\mathrm{SL}(2,\mathbb{R})$ with twisted conjugacy classes in $\times\,\mathrm{SU}(2)\times\mathrm{SU}(2)$,

$$\mathcal{C}_{\mathrm{SL}(2,\mathbb{R})}r_1\times\mathcal{C}^{\tilde{T}}_{\mathrm{SU}(2)\times\mathrm{SU}(2)}\tilde{r}\,.$$

We have studied in some detail the twisted conjugacy classes of $\mathrm{SU}(2)\times\mathrm{SU}(2)$, showing that they can be characterized as the inverse images, under the group mul-

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tiplication, of the standard conjugacy classes of SU(2). In particular, they consist of two totally-geodesic 3-spheres and a family of homologically trivial but homotopically non-trivial five-dimensional submanifolds diffeomorphic to $S^2 \times S^3$. These solutions give rise to even-dimensional branes embedded in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ and odd-dimensional D-branes wrapped on the flat S^1, and it is again the odd-dimensional branes that preserve half of the space-time supersymmetry, as expected.

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